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Abstract. A space $X$ is said to have property (USC) (resp. (LSC)) if whenever $\{f_n : n \in \omega\}$ is a sequence of upper (resp. lower) semicontinuous functions from $X$ into the closed unit interval $[0, 1]$ converging pointwise to the constant function 0 with the value 0, there is a sequence $\{g_n : n \in \omega\}$ of continuous functions from $X$ into $[0, 1]$ such that $f_n \leq g_n \ (n \in \omega)$ and $\{g_n : n \in \omega\}$ converges pointwise to 0. In this paper, we study spaces having these properties and related ones. In particular, we show that (a) for a subset $X$ of the real line, $X$ has property (USC) if and only if it is a $\sigma$-set; (b) if $X$ is a space of non-measurable cardinal and has property (LSC), then it is discrete. Our research comes of Scheepers’ conjecture on properties of non-$V$ measurable cardinal and has property $Q_z S_q R$ then it is discrete. W

1. Introduction and preliminaries

In this paper all spaces are assumed to be Tychonoff. We denote by $\mathbb{I}$ the closed unit interval $[0, 1]$. The symbol 0 is the constant function with the value 0, and for a sequence $\{f_n : n \in \omega\}$ of real-valued functions on a set $X$, the symbol $f_n \rightarrow 0$ means that $\{f_n : n \in \omega\}$ converges pointwise to 0 (i.e. $f_n(x) \rightarrow 0$ for every $x \in X$). Recall that a real-valued function $f$ on a space $X$ is upper (resp. lower) semicontinuous if for every real number $r$ the set $\{x \in X : f(x) < r\}$ (resp. $\{x \in X : f(x) > r\}$) is open in $X$. In this paper we study spaces having the properties defined below.

Definition 1.1. A space $X$ has property (USC) (resp. (LSC)) if for every sequence $\{f_n : n \in \omega\}$ of upper (resp. lower) semicontinuous functions from $X$ into $\mathbb{I}$ such that $f_n \rightarrow 0$, there is a sequence $\{g_n : n \in \omega\}$ of continuous functions from $X$ into $\mathbb{I}$ such that $f_n \leq g_n \ (n \in \omega)$ and $g_n \rightarrow 0$. A space $X$ has property (USC)$_s$ (resp. (LSC)$_s$) if for every sequence $\{f_n : n \in \omega\}$ of upper (resp. lower) semicontinuous functions from $X$ into $\mathbb{I}$ such that $f_n \rightarrow 0$, there are a subsequence $\{n_j : j \in \omega\}$ of $\omega$ and a sequence $\{g_j : j \in \omega\}$ of continuous functions from $X$ into $\mathbb{I}$ such that $f_{n_j} \leq g_j \ (j \in \omega)$ and $g_j \rightarrow 0$. A space $X$ property (USC)$_m$ (resp. (LSC)$_m$) if for every sequence $\{f_n : n \in \omega\}$ of upper (resp. lower) semicontinuous functions from $X$ into $\mathbb{I}$ such that $f_n \rightarrow 0$ and $f_{n+1} \leq f_n \ (n \in \omega)$, there is a sequence $\{g_n : n \in \omega\}$ of continuous functions from $X$ into $\mathbb{I}$ such that $f_n \leq g_n \ (n \in \omega)$ and $g_n \rightarrow 0$.

The implications (USC) $\rightarrow$ (USC)$_s$ $\rightarrow$ (USC)$_m$ hold by the definitions, and both arrows cannot be reversed even for subspaces of the real line (see Example 2.16 (1), Corollaries 2.15 and 2.23). On the other hand, properties (LSC), (LSC)$_s$
and (LSC)$_m$ are equivalent as we shall show in Section 3. Now, we mention the motivation of our research. To do this, we recall some definitions and results from the literature.

**Definition 1.2.** A countable family $\{A_n : n \in \omega\}$ of subsets of a set $X$ is a $\gamma$-cover of $X$ [8] if every point $x \in X$ is contained in $A_n$ for all but finitely many $n \in \omega$. A space $X$ has property $S_1(\Gamma, \Gamma)$ [20] if for every sequence $\{U_n : n \in \omega\}$ of open $\gamma$-covers of $X$, there are $U_n \in U_m$ ($n \in \omega$) such that $\{U_n : n \in \omega\}$ is a $\gamma$-cover of $X$.

**Definition 1.3.** A sequence $\{f_n : n \in \omega\}$ of real-valued functions on a set $X$ converges quasi-normally to 0 [1] if there is a sequence $\{\varepsilon_n : n \in \omega\}$ of positive real numbers converging to 0 such that for each $x \in X$, $|f_n(x)| < \varepsilon_n$ holds for all but finitely many $n \in \omega$. A space $X$ has property wQN [2] if every sequence $\{f_n : n \in \omega\}$ of real-valued continuous functions on $X$ such that $f_n \to 0$ converges quasi-normally to 0, and $X$ has property wQN [2] if every sequence $\{f_n : n \in \omega\}$ of real-valued continuous functions on $X$ such that $f_n \to 0$ contains a subsequence which converges quasi-normally to 0.

For covering characterizations of properties QN and wQN, see [4] and [18, Theorems 2.5 and 3.7]. Obviously property QN implies wQN, and it is known that every perfectly normal space having property QN has $S_1(\Gamma, \Gamma)$ (see [3, Theorem 7] and [18, Theorem 3.7]). Scheepers [21] showed that every space having property $S_1(\Gamma, \Gamma)$ has wQN, and conjectured that the converse holds for perfectly normal spaces. By Dow’s result in [5] it is consistent with ZFC that property wQN coincides with QN, and hence, so is that Scheepers’ conjecture holds (see [3]). It, however, is still open if Scheepers’ conjecture holds in ZFC. Concerning this, Bukovský introduced the following notion which is stronger than property wQN.

**Definition 1.4** ([3]). A space $X$ has property wQN$^*$ if every sequence $\{f_n : n \in \omega\}$ of upper semicontinuous functions from $X$ into $I$ such that $f_n \to 0$, contains a subsequence which converges quasi-normally to 0.

Bukovský [3] showed that property $S_1(\Gamma, \Gamma)$ implies wQN$^*$, and recently, Sakai [19] showed that property wQN$^*$ implies $S_1(\Gamma, \Gamma)$. Thus, $S_1(\Gamma, \Gamma)$ and wQN$^*$ are equivalent. Hence, we have the following.

**Proposition 1.5.** Every space having properties both wQN and (USC)$_s$ has property $S_1(\Gamma, \Gamma)$.

**Proof.** Let $X$ be a space having properties both wQN and (USC)$_s$. It is enough to show that $X$ has property wQN$^*$. Let $\{f_n : n \in \omega\}$ be a sequence of upper semicontinuous functions from $X$ into $I$ such that $f_n \to 0$. By property (USC)$_s$, there are a subsequence $\{n_j : j \in \omega\}$ of $\omega$ and a sequence $\{g_j : j \in \omega\}$ of continuous functions from $X$ into $I$ such that $f_{n_j} \leq g_j$ ($j \in \omega$) and $g_j \to 0$. By property wQN, the sequence $\{g_j : j \in \omega\}$ contains a subsequence which converges quasi-normally to 0. This implies that the sequence $\{f_{n_j} : j \in \omega\}$ also has a subsequence which converges quasi-normally to 0. \(\Box\)

In view of Scheepers’ conjecture, it is natural to ask if every perfectly normal space having property wQN has (USC)$_s$. Although the authors cannot solve this, it is interesting to know what kind of spaces have property (USC)$_s$. Other properties are natural byproducts occurred from this motivation.
Throughout this paper, $\mathbb{R}$ denotes the real line with the usual topology and $\omega$ denotes the first infinite ordinal. As usual, an ordinal is the set of smaller ordinals. Other terms and notation will be used as in [6] and [9].

2. Properties (USC), (USC)$_s$ and (USC)$_m$

A family $\{A_n : n \in \omega\}$ of subsets in a set $X$ is said to be point-finite if $\{n \in \omega : x \in A_n\}$ is finite for each $x \in X$. A subset $Z$ in a space $X$ is called a zero-set if $Z = f^{-1}(0)$ for some continuous function $f : X \to \mathbb{I}$. The complement of a zero-set is called a cozero-set.

Lemma 2.1. Let $\{f_n : n \in \omega\}$ be a sequence of functions from a space $X$ into $\mathbb{I}$ and let $F_{n,k} = \{x \in X : f_n(x) \geq 2^{-(k+1)}\}$ for each $n \in \omega$ and $k \in \omega$. Assume that, for every $k \in \omega$, there exist point-finite families $\{Z_{n,k} : n \in \omega\}$ of zero-sets in $X$ and $\{U_{n,k} : n \in \omega\}$ of cozero-sets in $X$ such that $F_{n,k} \subset Z_{n,k} \subset U_{n,k}$ for each $n \in \omega$. Then there exists a sequence $\{g_n : n \in \omega\}$ of continuous functions from $X$ into $\mathbb{I}$ such that $f_n \leq g_n (n \in \omega)$ and $g_n \to 0$.

Proof. For each $n \in \omega$ and $k \in \omega$, there is a continuous function $g_{n,k}$ on $X$ such that $0 \leq g_{n,k} \leq 2^{-(k+1)}$; $g_{n,k}(X \setminus U_{n,k}) = \{0\}$ and $g_{n,k}(Z_{n,k}) = \{2^{-(k+1)}\}$. We put $g_n = \sum_{k \in \omega} g_{n,k}$ for each $n \in \omega$. Then each $g_n$ is a continuous function from $X$ into $\mathbb{I}$. Let $x \in X$ be fixed. We show that $f_n(x) \leq g_n(x)$ for each $n \in \omega$. If $f_n(x) = 0$, then clearly $f_n(x) \leq g_n(x)$. So we may assume that $2^{-(k+1)} \leq f_n(x) \leq 2^{-k}$ for some $k \in \omega$. Then $x \in F_{n,j}$ for each $j \geq k$, which implies that

$$g_n(x) = \sum_{j \geq k} g_{n,j}(x) = \sum_{j \geq k} 2^{-(j+1)} = 2^{-k} \geq f_n(x).$$

Next, to show that $g_n(x) \to 0$ ($n \to \infty$), let $\varepsilon > 0$ and choose $k \in \omega$ with $2^{-k} < \varepsilon$. Then, for each $j < k$, there exists $m_j \in \omega$ such that $x \not\in U_{n,j}$ for each $n \geq m_j$. Put $m = \max\{m_j : j < k\}$. If $n \geq m$, then $g_{n,j}(x) = 0$ for each $j < k$, which implies that

$$g_n(x) = \sum_{j \geq k} g_{n,j}(x) \leq \sum_{j \geq k} 2^{-(j+1)} = 2^{-k} < \varepsilon.$$ 

Hence, $g_n(x) \to 0$ ($n \to \infty$).

Theorem 2.2. The following statements are equivalent.

1. $X$ has property (USC);
2. For every point-finite family $\{F_n : n \in \omega\}$ of closed sets in $X$, there are point-finite families $\{Z_n : n \in \omega\}$ of zero-sets in $X$ and $\{U_n : n \in \omega\}$ of cozero-sets in $X$ such that $F_n \subset Z_n \subset U_n$ for each $n \in \omega$.

Proof. (1) $\Rightarrow$ (2). Let $\{F_n : n \in \omega\}$ be a point-finite family of closed sets in $X$. For each $n \in \omega$, consider the function $f_n$ on $X$ defined by $f_n[F_n] = \{1\}$ and $f_n[X \setminus F_n] = \{0\}$. Then each $f_n$ is upper semicontinuous and $f_n \to 0$. Hence there is a sequence $\{g_n : n \in \omega\}$ of continuous functions on $X$ satisfying $f_n \leq g_n (n \in \omega)$ and $g_n \to 0$. For each $n \in \omega$, we put

$$Z_n = \{x \in X : g_n(x) \geq 2/3\} \quad \text{and} \quad U_n = \{x \in X : g_n(x) > 1/3\}.$$ 

Then, $Z_n$ is a zero-set in $X$, $U_n$ is a cozero-set in $X$ and $F_n \subset Z_n \subset U_n$ for each $n \in \omega$. Since $g_n \to 0$, $\{U_n : n \in \omega\}$ is point-finite in $X$. 

(2) $\rightarrow$ (1). Let $\{f_n : n \in \omega\}$ be a sequence of upper semicontinuous functions from $X$ into $\mathbb{I}$ such that $f_n \to 0$. For each $n \in \omega$ and $k \in \omega$ we put

$$F_{n,k} = \{x \in X : f_n(x) \geq 2^{-(k+1)}\}.$$ 

Then each $F_{n,k}$ is closed in $X$ and $\{F_{n,k} : n \in \omega\}$ is point-finite in $X$ for each $k \in \omega$. Hence, by (2) and Lemma 2.1, we can find a sequence $\{g_n : n < \omega\}$ of continuous functions from $X$ into $\mathbb{I}$ such that $f_n \leq g_n$ $(n \in \omega)$ and $g_n \to 0$. □

**Corollary 2.3.** For a normal space $X$, the following statements are equivalent.

1. $X$ has property (USC);
2. For every point-finite family $\{F_n : n \in \omega\}$ of closed sets in $X$, there exists a point-finite family $\{U_n : n \in \omega\}$ of open sets in $X$ such that $F_n \subset U_n$ for each $n \in \omega$;
3. For every open $\gamma$-cover $\{U_n : n \in \omega\}$ of $X$, there exists a closed $\gamma$-cover $\{F_n : n \in \omega\}$ of $X$ such that $F_n \subset U_n$ for each $n \in \omega$.

**Proof.** The equivalence of (1) and (2) follows from Theorem 2.2 and the fact that every pair of disjoint closed sets in a normal space can be separated by a continuous function. Observe that a family $\{U_n : n \in \omega\}$ of subsets of $X$ is a $\gamma$-cover of $X$ if and only if $\{X \setminus U_n : n \in \omega\}$ is point-finite in $X$. Hence, (2) is equivalent to (3). □

A space $X$ satisfying the condition (2) in Corollary 2.3 is called $(\gamma, \gamma)$-shrinkable in [18]. Recall from [15] that a space $X$ is a $\sigma$-set (or sometimes, $\sigma$-space) if every $F_\sigma$-set in $X$ is a $G_\delta$-set in $X$. For instance, a Sierpiński set is a $\sigma$-set [15, Theorem 4.1]. It was proved in [18, Corollary 3.4] that a perfectly normal space $X$ is $(\gamma, \gamma)$-shrinkable if and only if it is a $\sigma$-set. Hence, we have the following.

**Corollary 2.4.** For a perfectly normal space $X$, $X$ has property (USC) if and only if it is a $\sigma$-set.

Haleš [10] showed that every perfectly normal $\sigma$-set having property wQN has $S_1(\Gamma, \Gamma)$. This immediately follows from Proposition 1.5 and Corollary 2.4. The reader might ask if every perfectly normal space having property wQN is a $\sigma$-set. It is known that there is a counterexample under CH to this question (see Example 2.16). Haleš also proved in [10, Theorem 7] that every perfectly normal space having property QN is a $\sigma$-set (for metrizable spaces, this is due to Reclaw [16]; see also [18, Theorem 3.5]). Hence, we have the following by Corollary 2.4.

**Corollary 2.5.** Every perfectly normal space having property QN has property (USC).

A space $X$ is said to be scattered if every non-empty subset of $X$ has an isolated point in itself. For a space $X$ and an ordinal number $\alpha$, we define $X^{(\alpha)}$ as follows: $X^{(0)} = X$, $X^{(\alpha+1)}$ is the set of all non-isolated points of $X^{(\alpha)}$, and if $\alpha$ is a limit, then $X^{(\alpha)} = \bigcap\{X^{(\beta)} : \beta < \alpha\}$. Obviously a space $X$ is scattered if and only if $X^{(\alpha)} = \emptyset$ for some $\alpha$. For a scattered space $X$, we put $ht(X) = \min\{\alpha : X^{(\alpha)} = \emptyset\}$. Every subset of a scattered space is scattered, and the next lemma is immediate from the definitions.

**Lemma 2.6.** Let $X$ be a scattered space with $ht(X) = \alpha$. If $Y \subset X \setminus X^{(\beta)}$ and $\beta < \alpha$, then $ht(Y) \leq \beta$.

A space is said to be metacompact (or weakly paracompact [6]) if every open cover of it has a point-finite open refinement.
Theorem 2.7. Every normal metacompact scattered space has property (USC).

Proof. Let $X$ be a normal metacompact scattered space with $ht(X) = \alpha$, and assume that every normal metacompact scattered space $Y$ with $ht(Y) < \alpha$ has property (USC). Let $\{F_n : n \in \omega\}$ be a point-finite family of closed sets in $X$.

If $\alpha = \beta + 1$, then $\{F_n \cap X^{(\beta)} : n \in \omega\}$ is locally finite in $X$ since $X^{(\beta)}$ is closed and discrete in $X$. Since $X$ is metacompact, it follows from [6, 5.5.17(b)] that there is a point-finite family $\{U_n : n \in \omega\}$ of open sets in $X$ such that $F_n \cap X^{(\beta)} \subseteq U_n$ for each $n \in \omega$. Since $X$ is normal, there is a family $\{V_n : n \in \omega\}$ of open sets in $X$ such that $F_n \setminus U_n \subseteq V_n$ and $\bigcap_{\alpha} X^{(\beta)} = \emptyset$. Both normality and metacompactness are hereditary with respect to $F_n$-sets [6, 2.1.E.(a), 5.3.C.(b)]. Hence $\bigcup_{n \in \omega} V_n$ is normal metacompact scattered. By Lemma 2.6, $ht(\bigcup_{n \in \omega} V_n) \leq \beta$, hence $\bigcup_{n \in \omega} V_n$ has property (USC). Therefore there is a point-finite family $\{W_n : n \in \omega\}$ of open sets in $X$ such that $F_n \setminus U_n \subseteq W_n \subseteq V_n$ for each $n \in \omega$. The family $\{U_n \cup W_n : n \in \omega\}$ is point-finite and satisfies $F_n \subseteq U_n \cup W_n$ for each $n \in \omega$. By Corollary 2.3, $X$ has property (USC).

If $\alpha$ is limit, then $X = \bigcup \{X \setminus X^{(\beta)} : \beta < \alpha\}$. For each point $x \in X$, there are $\beta_x < \alpha$ and an open neighborhood $U_x$ of $x$ such that $U_x \cap X^{(\beta_x)} = \emptyset$. By Lemma 2.6, $ht(U_x) \leq \beta_x < \alpha$, hence $U_x$ has property (USC). Using metacompactness, we take a point-finite open refinement $\{V_x : s \in S\}$ of $\{U_x : x \in X\}$ in $X$ normal by property (USC). Therefore there is a point-finite family $\{W_x : s \in S\}$ of closed sets in $X$ such that $C_s \subseteq V_s$ for each $s \in S$. For each $s \in S$, $\{C_s \cap F_n : n \in \omega\}$ is point-finite and $\bigcup_{\alpha} V_x$ has property (USC). Therefore there is a point-finite family $\{W_{x,n} : n \in \omega\}$ of open sets in $X$ such that $C_s \cap F_n \subseteq W_{x,n} \subseteq V_s$ for each $n \in \omega$. Finally, let $W_n = \bigcup \{W_{x,n} : s \in S\}$ for each $n \in \omega$. Then $F_n \subseteq W_n (n \in \omega)$ and $\{W_n : n \in \omega\}$ is point-finite in $X$. By Corollary 2.3, $X$ has property (USC).

Corollary 2.8. Every compact scattered space has property (USC).

Corollary 2.9. Every ordinal with the order topology has property (USC).

Proof. Every successor ordinal has (USC) by Corollary 2.8. Every limit ordinal with countable cofinality has (USC), since it is homeomorphic to the topological sum of successor ordinals. Let $\alpha$ be a limit ordinal with uncountable cofinality. Then, for every point-finite family $\{F_n : n \in \omega\}$ of closed sets in $\alpha$, only finitely many $F_n$’s can unbounded in $\alpha$. By this fact, using Theorem 2.2 and Corollary 2.8, we can prove that $\alpha$ has property (USC).

We turn to considering property (USC)$_s$.

Theorem 2.10. The following statements are equivalent.

1. $X$ has property (USC)$_s$.

2. For every point-finite family $\{F_n : n \in \omega\}$ of closed sets in $X$, there are a subsequence $\{n_j : j \in \omega\}$ of $\omega$, point-finite families $\{Z_{n_j} : j \in \omega\}$ of zero-sets in $X$ and $\{U_{n_j} : j \in \omega\}$ of cozero-sets in $X$ such that $F_{n_j} \subseteq Z_{n_j} \subseteq U_{n_j}$ for each $j \in \omega$.

Proof. The implication (1) $\rightarrow$ (2) can be proved by the same arguments as in Theorem 2.2.

(2) $\rightarrow$ (1). Let $\{f_n : n \in \omega\}$ be a sequence of upper semicontinuous functions from $X$ into $[0, 1]$ such that $f_n \rightarrow 0$. For each $n \in \omega$ and $k \in \omega$, we put

$$F_{n,k} = \{x \in X : f_n(x) \geq 2^{-(k+1)}\}.$$
Then $F_{n,k}$ is closed in $X$ and $\{F_{n,k} : n \in \omega\}$ is point-finite in $X$ for each $k \in \omega$. We show that there are a subsequence $\{n_j : j \in \omega\}$, families $\{Z_{n_j,k} : j, k \in \omega\}$ of zero-sets in $X$ and $\{U_{n_j,k} : j, k \in \omega\}$ of cozero-sets in $X$ such that $\{Z_{n_j,k} : j \in \omega\}$ and $\{U_{n_j,k} : j \in \omega\}$ are point-finite for each $k \in \omega$ and $F_{n_j,k} \subset Z_{n_j,k} \subset U_{n_j,k}$ for each $j, k \in \omega$. By induction on $j \in \omega$, we can define, by applying (2), an infinite subset $A_j \subset \omega$ such that $A_j \subset A_{j-1}$, where $A_{-1} = \omega$, point-finite families $\{Z_{n,j} : n \in A_j\}$ of zero-sets in $X$ and $\{U_{n,j} : n \in A_j\}$ of cozero-sets in $X$ such that $F_{n,j} \subset Z_{n,j} \subset U_{n,j}$ for each $n \in A_j$. Put $n_j = \min A_j$ for each $j \in \omega$. We may assume that $n_j < n_{j+1}$ for each $j \in \omega$. Then $\{n_j : j \in \omega\}$ is a required subsequence of $\omega$. For each $k \in \omega$, the sets $Z_{n_j,k}$ and $U_{n_j,k}$ have already been defined for all $j \geq k$. For $j < k$, we put $Z_{n_j,k} = U_{n_j,k} = X$. Then the families $\{Z_{n_j,k} : j \in \omega\}$ and $\{U_{n_j,k} : j \in \omega\}$ $(k \in \omega)$ are what we want. By Lemma 2.1, there is a sequence $\{g_j : j \in \omega\}$ of continuous functions from $X$ into $I$ such that $f_{n_j} \leq g_j$ $(j \in \omega)$ and $g_j \to 0$.

Similarly to Corollary 2.3, we have the following by the preceding theorem.

**Corollary 2.11.** For a normal space $X$, the following statements are equivalent.

1. $X$ has property (USC)$_\omega$;
2. For every point-finite family $\{F_n : n \in \omega\}$ of closed sets in $X$, there exist a subsequence $\{n_j : j \in \omega\}$ of $\omega$ and a point-finite family $\{U_{n_j} : j \in \omega\}$ of open sets in $X$ such that $F_{n_j} \subset U_{n_j}$ for each $j \in \omega$;
3. For every open $\gamma$-cover $\{U_n : n \in \omega\}$ of $X$, there exist a subsequence $\{n_j : j \in \omega\}$ of $\omega$ and a closed $\gamma$-cover $\{F_{n_j} : j \in \omega\}$ of $X$ such that $F_{n_j} \subset U_{n_j}$ for each $j \in \omega$.

According to Hales, a space $X$ is called a $\gamma\gamma$-space if every open $\gamma$-cover of $X$ has a refinement which is a $\gamma$-cover consisting of clopen sets in $X$. Hales noted in [10, Lemma 1] that every $\gamma\gamma$-space having property wQN has $S_1(\Gamma, \Gamma)$. This result immediately follows from Proposition 1.5 and Theorem 2.10.

Since every cozero-set is the union of a countable point-finite family consisting of zero-sets (see [18, Lemma 3.2]), every $F_\sigma$-set of a perfectly normal space is the union of a countable point-finite family consisting of zero-sets. Therefore a perfectly normal space $X$ is a $\sigma$-set if and only if for every point-finite family $\{F_n : n \in \omega\}$ of closed sets in $X$, $\bigcup_{n \in \omega} F_n$ is a $G_\delta$-set in $X$. In a perfectly normal space, property (USC)$_\omega$ has a characterization similar to a $\sigma$-set.

**Corollary 2.12.** For a perfectly normal space $X$, the following statements are equivalent.

1. $X$ has property (USC)$_\omega$;
2. For every point-finite family $\{F_n : n \in \omega\}$ of closed sets in $X$, there is a subsequence $\{n_j : j \in \omega\}$ of $\omega$ such that $\bigcup_{j \in \omega} F_{n_j}$ is a $G_\delta$-set in $X$.

**Proof.** (1) $\to$ (2). Let $\{F_n : n \in \omega\}$ be a point-finite family of closed sets in $X$. Take a subsequence $\{n_j : j \in \omega\}$ of $\omega$ and a point-finite family $\{U_j : j \in \omega\}$ of open sets in $X$ such that $F_{n_j} \subset U_j$ for each $j \in \omega$. For each $j \in \omega$, let $F_{n_j} = \bigcap_{k \in \omega} G_{j,k}$, where each $G_{j,k}$ is open in $X$, $G_{j,0} = U_j$ and $G_{j,k+1} \subset G_{j,k}$ $(k \in \omega)$. Then $\bigcup_{j \in \omega} F_{n_j} = \bigcap_{k \in \omega} \bigcup_{j \in \omega} G_{j,k}$.

(2) $\to$ (1). Let $\{F_{n} : n \in \omega\}$ be a point-finite family of closed sets in $X$. Using the statement (2) inductively, we can take a decreasing sequence $\{A_k : k \in \omega\}$ of infinite subsets of $\omega$ such that $a_n > n_{k+1}$, where $n_k = \min A_k$ $(k \in \omega)$, and (b) for each
$k \in \omega$, $\bigcup_{n \in A_k} F_n$ is a $G_\delta$-set in $X$. For each $k \in \omega$, let $\bigcup_{n \in A_k} F_n = \bigcap_{j \in \omega} G_{k,j}$, where each $G_{k,j}$ is open in $X$ and $G_{k,j+1} \subset G_{k,j}$ $(j \in \omega)$. Without loss of generality, we may assume that for each $j \in \omega$ the sequence $\{G_{k,j} : k \in \omega\}$ is also decreasing. It is easy to check that $\{G_{k,k} : k \in \omega\}$ is point-finite in $X$ and $F_{m_k} \subset G_{k,k}$ for each $k \in \omega$. By Corollary 2.11, $X$ has property (USC)$_s$.

**Definition 2.13.** A family $A$ of subsets of a set $X$ is called an $\omega$-cover of $X$ [8] if every finite subset in $X$ is contained in some member of $A$. A space $X$ has property $(\gamma)$ [8] if for each open $\omega$-cover $U$ of $X$ contains a $\gamma$-cover, i.e., there exists a sequence $\{U_n : n \in \omega\} \subset \mathcal{U}$ with $X = \bigcup_{n \in \omega} \bigcap_{m \geq n} U_m$. A space $X$ has property $(\kappa)$ [17] if every pairwise disjoint sequence $\{F_n : n \in \omega\}$ of finite subsets in $X$ there are a subsequence $\{n_j : j \in \omega\}$ of $\omega$ and a point-finite family $\{U_j : j \in \omega\}$ of open sets in $X$ such that $F_{n_j} \subset U_j$ for each $j \in \omega$.

For a space $X$, we denote by $C_p(X)$ the space of all real-valued continuous functions on $X$ with the topology of pointwise convergence. Gerlits and Nagy [8] proved that $C_p(X)$ is Fréchet-Urysohn if and only if $X$ has property $(\gamma)$, and Sakai [17] proved that $C_p(X)$ is $\kappa$-Fréchet-Urysohn if and only if $X$ has property $(\kappa)$.

**Proposition 2.14.** Property $(\gamma)$ implies property $(USC)_s$, and property $(USC)_s$ implies property $(\kappa)$.

**Proof.** Let $X$ be a space having property $(\gamma)$, and let $\{F_n : n \in \omega\}$ be a point-finite family of closed sets in $X$. Then $X$ is Lindelöf, and hence, normal by the definition. For each finite set $A$ in $X$, take $n(A) \in \omega$ and an open set $U(A)$ in $X$ such that $A \subset U(A)$ and $U(A) \cap F_{n(A)} = \emptyset$. Then $U(A) : A$ is finite in $X$ is an $\omega$-cover of $X$. We take a $\gamma$-cover $\{U(A_k) : k \in \omega\}$. Then $\{X \setminus U(A_k) : k \in \omega\}$ is point-finite and $F_{n(A_k)} \subset X \setminus U(A_k)$ for each $k \in \omega$. By choosing a subsequence if necessary, we can assume that $n(A_k) \neq n(A_{k'})$ whenever $k \neq k'$. Hence, by Corollary 2.11, $X$ has property $(USC)_s$. The latter part follows from their definitions. □

A separable metrizable space is said to be always of the first category if every dense-in-itself subset of it is of the first category in itself [13, p. 516]. Since every separable metrizable space having property $(\kappa)$ is always of the first category [17, Theorem 3.2], we have the following.

**Corollary 2.15.** Every separable metrizable space having property $(USC)_s$ is always of the first category.

**Example 2.16.** (1) Under CH, Galvin and Miller [7] constructed a subset of $\mathbb{R}$ having property $(\gamma)$ which is not a $\lambda$-set (hence, not a $\sigma$-set) (see [17, Example 3.9 (1)]). By Corollary 2.4 and Proposition 2.14, this set has property $(USC)_s$, but does not have property $(USC)_s$.

(2) Every Sierpiński set is a $\sigma$-set, and hence, it has property (USC) by Corollary 2.4. On the other hand, every Sierpiński set can not have property $(\gamma)$ since every subspace of $\mathbb{R}$ with property $(\gamma)$ has strong measure zero [8]. Let $\alpha$ be an uncountable ordinal with uncountable cofinality. Then, $\alpha$ has property (USC) by Corollary 2.9, but does not have property $(\gamma)$ since every space with $(\gamma)$ must be Lindelöf by the definition.

**Lemma 2.17.** Every Čech-complete space having property $(USC)_s$ is scattered.

**Proof.** Let $X$ be a Čech-complete space having property $(USC)_s$. Assume that $X$ is not scattered. Then by a standard argument, there is a compact set $K$ in $X$ and
a continuous map from $K$ onto the Cantor cube $\{0, 1\}^\omega$. Using property $(\text{USC})_s$ of $K$, it is easy to check that $\{0, 1\}^\omega$ has also property $(\text{USC})_s$. By Corollary 2.15, this is a contradiction. □

**Theorem 2.18.** Let $X$ be a normal metacompact Čech-complete space. Then the following are equivalent.

1. $X$ has property $(\text{USC})$;
2. $X$ has property $(\text{USC})_s$;
3. $X$ is scattered.

**Proof.** (1) $\rightarrow$ (2) always holds. (2) $\rightarrow$ (3) follows from Lemma 2.17. (3) $\rightarrow$ (1) follows from Theorem 2.7. □

By Theorem 2.18, the one-point compactification of a discrete space has property $(\text{USC})$, while the Čech-Stone compactification $\beta\omega$ does not have $(\text{USC})_s$. Gerlits and Nagy [8] proved that a Lindelöf Čech-complete space has property $(\gamma)$ if and only if it is scattered. Hence we have the following.

**Corollary 2.19.** Let $X$ be a Lindelöf Čech-complete space. Then the following are equivalent.

1. $X$ has property $(\text{USC})$;
2. $X$ has property $(\text{USC})_s$;
3. $X$ is scattered;
4. $X$ has property $(\gamma)$.

Finally we investigate property $(\text{USC})_m$. Unlike properties $(\text{USC})$ and $(\text{USC})_s$, spaces in much wider classes have property $(\text{USC})_m$.

**Definition 2.20 ([11]).** A real-valued function on a space is said to be **locally bounded** if each point of the space has a neighborhood on which the function is bounded. A space $X$ is called a **cb-space** if for each locally bounded function $f$ on $X$ there is a continuous function $g$ on $X$ such that $|f| \leq g$.

**Lemma 2.21 ([14]).** A space $X$ is a cb-space if and only if for every decreasing sequence $\{F_n : n \in \omega\}$ of closed sets in $X$ with empty intersection, there exists a decreasing sequence $\{Z_n : n \in \omega\}$ of zero-sets in $X$ with empty intersection such that $F_n \subseteq Z_n$ for each $n \in \omega$.

**Theorem 2.22.** A space $X$ has property $(\text{USC})_m$ if and only if it is a cb-space.

**Proof.** Assume that $X$ has property $(\text{USC})_m$, and let $\{F_n : n \in \omega\}$ be a decreasing sequence of closed sets in $X$ with empty intersection. We define functions $f_n$’s from $X$ into $I$ as follows: $f_n[F_n] = \{1\}$ and $f_n[X \setminus F_n] = \{0\}$. Obviously $f_n \rightarrow 0$, $f_{n+1} \leq f_n$ $(n \in \omega)$ and every $f_n$ is upper semicontinuous. Take a sequence $\{g_n : n \in \omega\}$ of continuous functions from $X$ into $I$ such that $f_n \leq g_n$ $(n \in \omega)$ and $g_n \rightarrow 0$. For each $n \in \omega$, we put

$$Z_n = \bigcap_{j \leq n} \{x \in X : g_j(x) \geq 1/2\}.$$ 

Obviously $F_n \subseteq Z_n$ $(n \in \omega)$, and $\{Z_n : n \in \omega\}$ is a decreasing sequence of zero-sets in $X$ with empty intersection. By Lemma 2.21, $X$ is a cb-space.

Conversely assume that $X$ is a cb-space, and let $\{F_n : n \in \omega\}$ be a sequence of upper semicontinuous functions from $X$ into $I$ such that $f_n \rightarrow 0$ and $f_{n+1} \leq f_n$
((n \in \omega)). \text{ Put } F_{n,k} = \{x \in X : f_n(x) \geq 2^{-(k+1)}\} \text{ for each } n \in \omega \text{ and } k \in \omega. \text{ Then, for each } k \in \omega, \{F_{n,k} : n \in \omega\} \text{ is a decreasing sequence of closed sets in } X \text{ with empty intersection. Since } X \text{ is a cb-space, using Lemma 2.21 we can take a decreasing sequence } \{Z_{n,k} : n \in \omega\} \text{ of zero-sets in } X \text{ with empty intersection such that } F_{n,k} \subset Z_{n,k} \text{ for each } n \in \omega. \text{ Since each } Z_{n,k} \text{ is a zero-set in } X, \text{ by a simple observation we can take a decreasing sequence } \{U_{n,k} : n \in \omega\} \text{ of cozero-sets in } X \text{ with empty intersection such that } Z_{n,k} \subset U_{n,k} \text{ for each } n \in \omega. \text{ Hence, by Lemma 2.1 there is a sequence } \{g_n : n \in \omega\} \text{ of continuous functions from } X \text{ into } I \text{ such that } f_n \leq g_n (n \in \omega) \text{ and } g_n \to 0. \hfill \Box

Horne [11] proved that every normal countably paracompact space is a cb-space, and that every cb-space is countably paracompact. Hence we obtain the following.

**Corollary 2.23.** Every normal countably paracompact space has property (USC)$_m$, and every space having property (USC)$_m$ is countably paracompact.

A space $X$ is called a $P$-space if every $G_\delta$-subset of it is open in $X$.

**Proposition 2.24.** Every $P$-space having property (USC)$_m$ has property (USC).

**Proof.** Let $X$ be a normal $P$-space having property (USC)$_m$, and let $\{F_n : n \in \omega\}$ be a point-finite family of closed sets in $X$. For each $n \in \omega$, we put $F'_n = \bigcup\{F_k : k \geq n\}$. Since $X$ is a $P$-space, each $F'_n$ is closed in $X$. Obviously $\{F'_n : n \in \omega\}$ is a decreasing sequence with empty intersection. By Theorem 2.22, there is a decreasing sequence $\{Z_n : n \in \omega\}$ of zero-sets in $X$ with empty intersection such that $F'_n \subset Z_n$ for each $n \in \omega$. Since a zero-set is a $G_\delta$-set, each $Z_n$ is open (and closed) in $X$. Thus $\{Z_n : n \in \omega\}$ is a point-finite family of clopen sets in $X$ such that $F_n \subset Z_n$ for each $n \in \omega$. By Theorem 2.2, $X$ has property (USC). The latter part follows from Corollary 2.23. \hfill \Box

**Example 2.25.** (1) Obviously, properties (USC), (USC)$_s$ and (USC)$_m$ are hereditary with respect to closed subsets. We show that these properties are not hereditary. Let $Y$ be a locally compact, scattered space which is not countably paracompact (for example, the space $\Psi = \omega \cup D$ described in [9, 5I]), and consider the one-point compactification $\bar{X}$ of $Y$. Then $X$ has property (USC) by Corollary 2.8, but $Y$ fails to have property (USC)$_m$ by Corollary 2.23. Hence, (USC), (USC)$_s$ and (USC)$_m$ are not hereditary. It will be worth noting that, by Corollary 2.4, property (USC) is hereditary in the realm of $\mathbb{R}$, i.e., if $Y \subset X \subset \mathbb{R}$ and $X$ has (USC), then $Y$ also has (USC). The authors do not know if property (USC)$_s$ is hereditary in the realm of $\mathbb{R}$. (2) We show that the space $\Psi = \omega \cup D$ satisfies the statement (2) in Corollary 2.3. Let $\{F_n : n \in \omega\}$ be a point-finite family of closed sets in $\Psi$. For each $n \in \omega$, let $U_n = F_n \cup \{k \in \omega : k > n\}$. Then $U_n$ is open in $\Psi$ and contains $F_n$. It is easy to see that $\{U_n : n \in \omega\}$ is point-finite. Thus in Corollary 2.3, normality is necessary. Moreover, since every subset of $\Psi$ is a $G_\delta$-set, perfect normality is necessary in Corollary 2.4.

The following diagram holds for subsets of $\mathbb{R}$, where AFC means “always of the first category”. In addition, it is proved in [12, Theorem 2.11] that every Sierpiński set has property $S_1(\Gamma, \Gamma)$. 
\[ (\gamma) \rightarrow S_1(\Gamma, \Gamma) \rightarrow \text{wQN} \]
\[ \text{Sierpiński set} \rightarrow \sigma\text{-set} = (\text{USC}) \rightarrow (\text{USC})_n \rightarrow (\kappa) \rightarrow \text{AFC} \]

Here, the implication \((\gamma) \rightarrow S_1(\Gamma, \Gamma)\) is noted in [20], \(S_1(\Gamma, \Gamma) \rightarrow \text{wQN}\) is proved in [21, Theorems 6, 10], \(\text{wQN} \rightarrow \text{AFC}\) is proved in [2, Corollary 4.3], and \(\lambda\text{-set} \rightarrow (\kappa) \rightarrow \text{AFC}\) is proved in [17, Theorem 3.2]. Finally, in view of Corollary 2.5, the following question naturally arises.

**Question 2.26.** Does every perfectly normal space having property \(\text{wQN}\) have property \((\text{USC})_n\)?

By Proposition 1.5, Scheepers conjecture holds if the answer to the above question were positive.

3. Properties (LSC), \((\text{LSC})_n\) and \((\text{LSC})_m\)

In contrast with properties (USC), \((\text{USC})_n\) and \((\text{USC})_m\), we show that properties (LSC), \((\text{LSC})_n\), and \((\text{LSC})_m\) are all equivalent.

**Theorem 3.1.** For a space \(X\), the following statements are equivalent.

1. \(X\) has property (LSC);
2. \(X\) has property \((\text{LSC})_n\);
3. \(X\) has property \((\text{LSC})_m\);
4. For every decreasing sequence \(\{U_n : n \in \omega\}\) of open sets in \(X\) with empty intersection, there exists a sequence \(\{Z_n : n \in \omega\}\) of zero-sets in \(X\) with empty intersection such that \(U_n \cap Z_n \neq \emptyset\) for each \(n \in \omega\);
5. \(X\) is a \(P\)-space and for every point-finite family \(\{U_n : n \in \omega\}\) of open sets in \(X\), there exists a point-finite family \(\{V_n : n \in \omega\}\) of closed sets in \(X\) such that \(U_n \cap V_n \neq \emptyset\) for each \(n \in \omega\).

**Proof.** The implications (1) \(\rightarrow\) (2) \(\rightarrow\) (3) are trivial.

(3) \(\rightarrow\) (4). Let \(\{U_n : n \in \omega\}\) be a decreasing sequence of open sets in \(X\) with empty intersection. For each \(n \in \omega\), define a function \(f_n : X \rightarrow \mathbb{I}\) by \(f_n[U_n] = \{1\}\) and \(f_n[X \setminus U_n] = \{0\}\). Obviously, \(f_n \rightarrow 0\), \(f_{n+1} \leq f_n\) \(n \in \omega\) and every \(f_n\) is lower semicontinuous. Take a sequence \(\{g_n : n \in \omega\}\) of continuous functions from \(X\) into \(\mathbb{I}\) such that \(f_n \leq g_n\) \(n \in \omega\) and \(g_n \rightarrow 0\). For each \(n \in \omega\), let \(Z_n = \{x \in X : g_n(x) \geq 1/2\}\). Then \(\{Z_n : n \in \omega\}\) is a family of zero-sets in \(X\) with empty intersection such that \(U_n \cap Z_n \neq \emptyset\) for each \(n \in \omega\).

(4) \(\rightarrow\) (5). First we show that \(X\) is a \(P\)-space. Let \(A\) be a \(G_\delta\)-set in \(X\) and let \(x\) be any point in \(A\). Take a closed \(G_\delta\)-set \(F\) in \(X\) satisfying \(x \in F \subseteq A\). Let \(F = \bigcap_{n \in \omega} G_n\), where each \(G_n\) is open in \(X\) and \(G_{n+1} \subseteq G_n\) \((n \in \omega)\). Since \(\{G_n \setminus F : n \in \omega\}\) is a descending sequence of open sets in \(X\) with empty intersection, there is a family \(\{Z_n : n \in \omega\}\) of zero-sets in \(X\) with empty intersection such that \(G_n \setminus F \subseteq Z_n\) for each \(n \in \omega\). Take \(n \in \omega\) with \(x \notin Z_n\). Then \((X \setminus Z_n) \cap G_n\) is an open set in \(X\) such that \(x \in (X \setminus Z_n) \cap G_n \subseteq F \subseteq A\). Thus \(A\) is open in \(X\).

Now let \(\{U_n : n \in \omega\}\) be a point-finite family of open sets in \(X\). For each \(n \in \omega\), we put \(U'_n = \bigcup_{k > n} U_k\). Then \(\{U'_n : n \in \omega\}\) is a decreasing sequence of open sets in \(X\) with empty intersection. Take a family \(\{Z_n : n \in \omega\}\) of zero-sets in \(X\) with empty intersection such that \(U'_n \subseteq Z_n\) for each \(n \in \omega\). Since each \(Z_n\) is a \(G_\delta\)-set, it
is clopen in $X$. Let $V_n = Z_0 \cap \cdots \cap Z_n$ for each $n \in \omega$. The family $\{V_n : n \in \omega\}$ is what we need.

(5) $\Rightarrow$ (1). Let $\{f_n : n \in \omega\}$ be a sequence of lower semicontinuous functions from $X$ into $\mathbb{I}$ such that $f_n \to 0$. For each $n \in \omega$ and $k \in \omega$, we put

$$U_{n,k} = \{x \in X : f_n(x) > 2^{-(k+2)}\}.$$

Then for each $k \in \omega$, $\{U_{n,k} : n \in \omega\}$ is a point-finite family of open sets in $X$. By (5) there exist clopen sets $V_{n,k} (n, k \in \omega)$ satisfying that $U_{n,k} \subseteq V_{n,k}$ and for each $k \in \omega$ $\{V_{n,k} : n \in \omega\}$ is point-finite in $X$. Since $\{x \in X : f_n(x) \geq 2^{-(k+1)}\} \subseteq U_{n,k} \subseteq V_{n,k}$ for each $n \in \omega$ and $k \in \omega$, it follows from Lemma 2.1 that there exists a sequence $\{g_n : n \in \omega\}$ of continuous functions from $X$ into $I$ such that $f_n \leq g_n$ ($n \in \omega$) and $g_n \to 0$.

A cardinal $\kappa$ is said to be measurable [9, 12.1] if a set $X$ of cardinal $\kappa$ admits a $\{0,1\}$-valued measure $\mu$ such that $\mu(X) = 1$ and $\mu(\{x\}) = 0$, where $\mu$ is defined on the family of all subsets of $X$ and countably additive. It is known in [9, 12.2] that a cardinal $\kappa$ is non-measurable if and only if every ultrafilter with the countable intersection property has non-empty intersection. The smallest measurable cardinal (if any exist) is strongly inaccessible [9, 12.6]. A filter is said to be free if it has empty intersection.

**Corollary 3.2.** Let $X$ be a space of non-measurable cardinal. Then $X$ has property ($LSC$) if and only if $X$ is discrete.

**Proof.** Assume that $X$ has property (LSC) and there is a non-isolated point $x$ in $X$. Let $\{U_\alpha : \alpha < \kappa\}$ be a pairwise disjoint family of non-empty open subsets in $X$ satisfying the conditions $x \notin \overline{U_\alpha}$ ($\alpha < \kappa$) and $x \in \bigcup_{\alpha < \kappa} U_\alpha$, where $\kappa$ is an infinite cardinal. For every open neighborhood $U$ of $x$, we set $F(U) = \{\alpha < \kappa : U \cap U_\alpha \neq \emptyset\}$. Then $\mathcal{F} = \{F(U) : U \text{ is an open neighborhood of } x\}$ is a free filter base on $\kappa$. Let $\mathcal{G}$ be a uniform filter over $\kappa$ containing $\mathcal{F}$. Suppose that there is a decreasing sequence $\{G_n : n \in \omega\} \subseteq \mathcal{G}$ with empty intersection. Then obviously $\bigcup_{\alpha \in G_n} U_\alpha : n \in \omega\}$ is a point-finite family of open sets in $X$, hence $\bigcup_{\alpha \in G_n} U_\alpha : n \in \omega\}$ is also point-finite by Theorem 3.1. Take some $k \in \omega$ and an open neighborhood $V$ of $x$ such that $V \cap \bigcup_{\alpha \in G_k} U_\alpha = \emptyset$. Then $F(V) \cap G_k = \emptyset$. Since $\mathcal{G}$ is a filter, this is a contradiction. Therefore $\mathcal{G}$ has the countable intersection property. Since $\kappa$ has a free ultrafilter $\mathcal{G}$ with the countable intersection property, it is measurable, hence the cardinal of $X$ is measurable. The converse is trivial.

**Example 3.3.** There is a non-discrete space having property (LSC). Assume that there is a measurable cardinal $\kappa$. Let $p$ be a free ultrafilter on $\kappa$ with the countable intersection property. We consider the subspace $X(p, \kappa) = \{p\} \cup D(\kappa)$ of $\beta D(\kappa)$, where $D(\kappa)$ is the discrete space with cardinal $\kappa$, and $\beta D(\kappa)$ is the Cech-Stone compactification of $D(\kappa)$. We see that $X(p, \kappa)$ has property (LSC). Let $\{P_n : n \in \omega\}$ be a decreasing sequence of open sets in $X(p, \kappa)$ with empty intersection. Since $p$ has the countable intersection property, $P_n \notin p$ (i.e. $p \notin P_n$) for all but finitely many $n \in \omega$. Thus $P_n$ is clopen in $X(p, \kappa)$ for all but finitely many $n \in \omega$. Thus $X(p, \kappa)$ has property (LSC). It is easy to see that $X(p, \kappa)$ has property (USC) too.

We observe that every function on $X(p, \kappa)$ is upper semicontinuous or lower semicontinuous. Indeed for a function $f$ on $X(p, \kappa)$, let $A = \{x \in D(\kappa) : f(x) \leq f(p)\}$ and $B = \{x \in D(\kappa) : f(x) \geq f(p)\}$. Since $p$ is a ultrafilter, $A \in p$ or $B \in p$. 
If $A \in p$ (resp. $B \in p$), then $f$ is upper (resp. lower) semicontinuous. Thus $X(p, \kappa)$ has the following stronger property: for every sequence $\{f_n : n \in \omega\}$ of functions from $X(p, \kappa)$ into $I$ such that $f_n \to 0$, there is a sequence $\{g_n : n \in \omega\}$ of continuous functions from $X(p, \kappa)$ into $I$ such that $f_n \leq g_n$ ($n \in \omega$) and $g_n \to 0$.

The space $X(p, \kappa)$ is extremally disconnected (i.e. the closure of every open set is open). But not every space with (LSC) is extremally disconnected. Let $Y = X(p, \kappa) \times \{0, 1\}$ and let $Z$ be the quotient space obtained by identifying the two points $(p, 0)$ and $(p, 1)$ in $Y$. Then the space $Z$ has property (LSC) but is not extremally disconnected.

**Question 3.4.** Does every space having property (LSC) have property (USC) or (USC)$_u$ or (USC)$_m$?

**References**


Faculty of Education, Shizuoka University, Ohyaa, Shizuoka 422-8529, Japan
E-mail address: echota@ipc.shizuoka.ac.jp

Department of Mathematics, Kanagawa University, Yokohama 221-8686, Japan
E-mail address: sakaim01@kanagawa-u.ac.jp