A correction to “A non-implication between fragments of Martin’s Axiom related to a property which comes from Aronszajn trees”

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Abstract
In the paper A non-implication between fragments of Martin’s Axiom related to a property which comes from Aronszajn trees [1], Proposition 2.7 is not true. To avoid this error and correct Proposition 2.7, the definition of the property $R_1, \mathfrak{R}_1$ is changed. In [1], all proofs of lemmas and theorems but Lemma 6.9 are valid about this definition without changing the proofs. We give a new statement and a new proof of Lemma 6.9.

In the paper A non-implication between fragments of Martin’s Axiom related to a property which comes from Aronszajn trees [1], Proposition 2.7 is not true. For example, $T$ is an Aronszajn tree, $t_1$ and $t_3$ are incomparable node of $T$ in a model $N$, $t_2$ is a node of $T$ such that $t_2 \notin N$ and $t_2 <_T t_2$, $\sigma := \{t_2, t_3\}$ (which is in $a(T)$) and $I$ be an uncountable subset of $a(T)$ which forms a $\Delta$-system with root $\{t_1, t_3\}$. Then $\sigma \cap N = \{t_1\} \subseteq \{t_1, t_3\}$, but every element of $I$ is incompatible with $\sigma$ in $a(T)$.

To avoid this error and correct Proposition 2.7, the definition of the property $R_1, \mathfrak{R}_1$ is changed as follows.

Theorem 2.6. A forcing notion $Q$ in $\mathsf{FSCO}$ has the property $R_1, \mathfrak{R}_1$ if for any regular cardinal $\kappa$ larger than $\mathfrak{R}_1$, countable elementary submodel $N$ of $H(\kappa)$ which has the set $\{Q\}$, $I \in [I]^{\mathfrak{R}_1} \cap N$ and $\sigma \in Q \setminus N$, if $I$ forms a $\Delta$-system with root (exactly) $\sigma \cap N$, then there exists $I' \in [I]^{\mathfrak{R}_1} \cap N$ such that every member of $I'$ is compatible with $\sigma$ in $Q$.

Similarly, we should also change Proposition 2.8 and Proposition 2.10.2 as follows.

Proposition 2.8. The property $R_1, \mathfrak{R}_1$ is closed under finite support products in the following sense.

If $\{Q_\xi; \xi \in \Sigma\}$ is a set of forcing notions in $\mathsf{FSCO}$ with the property $R_1, \mathfrak{R}_1$, $\kappa$ is a large enough regular cardinal, $N$ is a countable elementary submodel of $H(\kappa)$ which has the set $\{Q_\xi\}$, $I \in [I]^{\mathfrak{R}_1} \cap N$ and $\sigma \in \bigcup_{\xi \in \Sigma} Q_\xi \setminus N$, if $I$ forms a $\Delta$-system with root (exactly) $\sigma \cap N$, then there exists $I' \in [I]^{\mathfrak{R}_1} \cap N$ such that every member of $I'$ is compatible with $\sigma$ in $\bigcup_{\xi \in \Sigma} Q_\xi$.
\( H(\kappa) \) which has the set \( \{ \{ Q_\xi; \xi \in \Sigma \} \} \). \( I \) is an uncountable subset of the finite support product \( \prod_{\xi \in \Sigma} Q_\xi \) in \( N \), \( \vec{\sigma} \in \prod_{\xi \in \Sigma} Q_\xi \setminus N \), \( I \) forms a \( \Delta \)-system with root (exactly) \( \vec{\sigma} \cap N \), that is,

- the set \( \{ \text{supp}(\vec{\tau}); \tau \in I \} \) forms a \( \Delta \)-system with root (exactly) \( \text{supp}(\vec{\sigma}) \cap N \), where \( \text{supp}(\vec{\tau}) := \{ \xi \in \Sigma; \vec{\tau}(\xi) \neq \emptyset \} \),

- for each \( \xi \in \text{supp}(\vec{\sigma}) \cap N \), the set \( \{ \vec{\tau}(\xi); \tau \in I \} \) forms a \( \Delta \)-system with root (exactly) \( \vec{\sigma}(\xi) \cap N \).

then there exists \( I' \in [I]^{|\kappa_1|} \cap N \) such that every element of \( I' \) is compatible with \( \vec{\sigma} \) in \( \prod_{\xi \in \Sigma} Q_\xi \).

**Proposition 2.10.2.** Let \( Q \) be a forcing notion in FSCO with the property \( R_{1, \kappa_1} \). Suppose that \( \kappa \) is a regular cardinal larger than \( \kappa_1 \), \( N \) is a countable elementary submodel of \( H(\kappa) \) which has the set \( \{ Q_i; i \in n \} \) is a finite sequence of members of the set \( \{ Q \}^{|\kappa_1|} \cap N \), and \( \sigma \in Q \setminus N \) such that the union \( \bigcup_{i \in n} I_i \) forms a \( \Delta \)-system with root (exactly) \( \sigma \cap N \).

Then there exists \( \langle \tau_i; i \in n \rangle \in \prod_{i \in n} I_i \) such that there exists a common extension of \( \sigma \) and the \( \tau_i \) in \( Q \).

The new definition of the property \( R_{1, \kappa_1} \) is less restrictive. All examples in the paper [1] have this property. In [1], all proofs of lemmas and theorems but Lemma 6.9 are valid about this definition without changing the proofs. For example, in the proof of Proposition 2.7, we have only to check for an uncountable subset \( I \) of \( \text{a}(\mathbb{P}) \) in a countable elementary submodel \( N \) of \( H(\kappa) \) and \( \sigma \in \text{a}(\mathbb{P}) \setminus N \) such that \( I \) forms a \( \Delta \)-system with root \( \sigma \cap N \). The proof of this proposition is completely same to the one in [1]. The proofs of Theorems 5.3 and 5.4 are adapted for this new definition. Because the property \( R_{1, \kappa_1} \) are applied for uncountable sets which form \( \Delta \)-systems with root exact “\( \tau \cap N \)” in the proofs of Theorems 5.3 and 5.4 in [1]. We apply the new Proposition 2.10.2 to these \( \Delta \)-systems.

We have to change only the statement and the proof of Lemma 6.9 as follows.

**Lemma 6.9.** Suppose that \( Q \) is a forcing notion in FSCO with the property \( R_{1, \kappa_1} \), \( I \) is an uncountable subset of \( Q \) such that

- \( I \) forms a \( \Delta \)-system with root \( \epsilon \), and

- for every \( \sigma \) and \( \tau \) in \( I \), either \( \max(\sigma \setminus \epsilon) < \min(\tau \setminus \epsilon) \) or \( \max(\tau \setminus \epsilon) < \min(\sigma \setminus \epsilon) \),

\( \vec{M} = (M_\alpha; \alpha \in \omega_1) \) is a sequence of countable elementary submodels of \( H(\kappa_2) \) such that \( \{ Q, I \} \in M_0 \), and for every \( \alpha \in \omega_1 \), \( \langle M_\beta; \beta \in \alpha \rangle \in M_\alpha \), and \( S \subseteq \omega_1 \setminus \{ 0 \} \) is stationary.

Then \( Q(Q, I, \vec{M}, S) \) is \( (T, S) \)-preserving.
Let $Q, I, M, S$ be as in the assumption of the statement of the lemma, and $T, \theta, N$ as in the statement of the definition of the $(T, S)$-preservation, moreover we suppose $M \in N$, to calculate levels of conditions in $Q$ and $(h, f) \in Q(Q, I, M, S) \cap N$. Suppose that $\omega_1 \cap N \not\in S$, because if $\omega_1 \cap N \in S$, then the condition $(h \cup \{\langle \omega_1 \cap N, \omega_1 \cap N \rangle\}, f)$ is as desired.

Let
\[
\delta := \sup \{ F(\omega_1 \cap N) + 1; F \in (\omega_1 \cap N) \}.
\]
Since $N$ is countable, $\delta$ is a countable ordinal. We will show that the condition $(h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f)$ of $Q(Q, I, M, S)$ is our desired one.

By Lemma 6.6 (in the original paper [1]), $(h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f)$ is $(N, Q(Q, I, M, S))$-generic. Suppose that $x \in T$ of height $\omega_1 \cap N$ such that for any subset $A \subseteq N$ of $T$, if $x \in A$, then there is $y \in A$ such that $y < x$. Let $A \in N$ is a $Q(Q, I, M, S)$-name for a subset of $T$. We will show that

\[
\langle h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f \rangle \Vdash Q^{\omega_1 \cap N} x \not\in A \text{ or } \exists y \in A \langle y < x \rangle.
\]
Let $\langle h', f' \rangle \leq Q(Q, I, M, S)$ $\langle h \cup \{\langle \omega_1 \cap N, \delta \rangle\}, f \rangle$, and assume that

\[
\langle h', f' \rangle \not\Vdash Q^{\omega_1 \cap N} x \not\in A \text{ "}.
\]
By strengthening $\langle h', f' \rangle$ if necessary, we may assume that

\[
\langle h', f' \rangle \Vdash Q^{\omega_1 \cap N} x \in A \text{ "}.
\]
We note that $\langle h' \cap N, f' \cap N \rangle$ is in $N$ (because $\omega_1 \cap N \in \text{dom}(h')$) and for every $\sigma \in \text{dom}(f') \setminus N$, $\min(\sigma \setminus \epsilon) > \delta$ by the definition of $Q(Q, I, M, S)$. Let

\[
L := \{ f'(\sigma); \sigma \in \text{dom}(f') \land f'(\sigma) \in \omega_1 \cap N \},
\]
which is a finite subset of $N$, hence is in $N$. For each $\alpha \in L$, let

\[
\tau_\alpha := \bigcup (\langle f'(\alpha) \rangle^{-1}\{\alpha\})
\]
Then $\langle \tau_\alpha; \alpha \in L \rangle$ is a condition of the product $L \cap Q$ and for each $\alpha \in L$, $\tau_\alpha$ is an extension of all members of $\langle f'(\alpha) \rangle^{-1}\{\alpha\}$ in $Q$. The sequence $\langle \tau_\alpha; \alpha \in L \rangle$ does not belong to $N$, however we notice that the sequence $\langle \tau_\alpha \cap N; \alpha \in L \rangle$ belongs to $N$. We define a function $F$ with the domain

\[
\{ t \in T; ht_T(t) > \max(\text{dom}(h' \cap N)) \}
\]
such that for each $t \in T$ of height larger than $\max(\text{dom}(h' \cap N))$,

\[
F(t) := \sup \{ \beta \in \omega_1; \text{ there exists } (k, g) \in Q(Q, I, M, S) \text{ such that }
\]

- $\text{min}(\text{dom}(k)) = ht_T(t)$,
- $k(\text{ht}_T(t)) = \beta$,
- $\langle h' \cap N \cup k, (f' \cap N) \cup g \rangle$ is a condition of $Q(Q, I, M, S)$,
- $\langle h' \cap N \cup k, (f' \cap N) \cup g \rangle \Vdash Q(Q, I, M, S) \langle t \in A \rangle$,
- for all $\alpha \in L$, $\text{min} \left( \bigcup \langle g^{-1}\{\alpha\} \setminus \epsilon \rangle \right) \geq \beta
\}$. 

\[
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\]
Then $F$ belongs to $N$. Let
\[ B := \{ t \in T; \text{ht}_T(t) > \max(\text{dom}(h' \upharpoonright N)) \& F(t) = \omega_1 \}, \]
which is also in $N$. We define a function $F'$ with the domain
\[ \max(\text{dom}(h' \upharpoonright N)) + 1, \omega_1 \]
such that for a countable ordinal $\beta$ larger than $\max(\text{dom}(h' \upharpoonright N))$,\[ F'(\beta) := \sup \{ F(t) + 1; t \in T \setminus B \& \text{ht}_T(t) \in (\max(\text{dom}(h' \upharpoonright N)), \beta) \}. \]
This $F'$ is a function from $\omega_1$ into $\omega_1$ and also in $N$. Hence $F'(\omega_1 \cap N) < \delta$ by the definition of $\delta$. Since letting $k = h' \setminus (h' \upharpoonright N)$ and $g = f' \setminus (f' \upharpoonright N)$, $\text{ht}_T(x) = h'(\omega_1 \cap N) = \delta$, $(h' \upharpoonright N) \cup k, (f' \upharpoonright N) \cup g \models_{Q(I, M, S)} \" x \in A \"$ and $\min (\bigcup \{ g^{-1}[\alpha] \setminus \epsilon \}) \geq \delta, F(x) \geq \delta$ holds. Therefore $x$ have to belong to $B$.
Thus by our assumption, there exists $y \in B$ such that $y < x$.
Since $F(y) = \omega_1$ and both $F$ and $y$ belong to $N$, there exists an uncountable subset $\{ \langle k_\xi, g_\xi \rangle; \xi \in \omega_1 \}$ of $Q(Q, I, M, S)$ such that for each $\xi$ and $\eta$ in $\omega_1$ with $\xi < \eta$,
- $\langle (h' \upharpoonright N) \cup k_\xi, (f' \upharpoonright N) \cup g_\xi \rangle$ is a condition of $Q(Q, I, M, S)$,
- $\langle (h' \upharpoonright N) \cup k_\xi, (f' \upharpoonright N) \cup g_\xi \rangle \models_{Q(Q, I, M, S)} \" y \in A \\"$,
- for all $\alpha \in L$,
\[
\max (\tau_\alpha \cap N) < \min \left( \bigcup \{ g_\xi^{-1}\{\alpha\} \setminus \epsilon \} \right)
< \max \left( \bigcup \{ g_\eta^{-1}\{\alpha\} \setminus \epsilon \} \right) < \min \left( \bigcup \{ g_\eta^{-1}\{\alpha\} \setminus \epsilon \} \right).
\]
For each $\xi \in \omega_1$ and $\alpha \in L$, let
\[
\mu_{\xi, \alpha} := \bigcup \{ (f' \upharpoonright N)^{-1}\{\alpha\} \cup g_\eta^{-1}\{\alpha\} \}.
\]
Then for every $\alpha \in L$, since
\[
\tau_\alpha \cap N = \bigcup \{ (f' \upharpoonright N)^{-1}\{\alpha\} \}
\]
(because of the assumption of $I$), the set $\{ \mu_{\xi, \alpha}; \xi \in \omega_1 \}$ forms a $\Delta$-system with root $\tau_\alpha \cap N$. So by the property $R_{I, R_{I}}$ of $LQ$ of Proposition 2.8, there exists $J'' \in [\omega_1]^{\aleph_1} \cap N$ such that every member of the set $\{ \langle \mu_{\xi, \alpha}; \alpha \in L \rangle; \xi \in J'' \}$ is compatible with $\langle \tau_\alpha; \alpha \in L \rangle$ in $LQ$. Therefore when we take any $\xi \in J'' \cap N$, for every $\alpha \in L$, $\mu_{\xi, \alpha} \cup \tau_\alpha$ is an extension of all members of $(f')^{-1}\{\alpha\} \cup g_\xi^{-1}\{\alpha\}$ in $Q$, so $(h' \cup k_\xi, f' \cup g_\xi)$ is a common extension of $(h', f')$ and $(k_\xi, g_\xi)$ in $Q(Q, I, M, S)$. Moreover it follows that
\[ (h' \cup k_\xi, f' \cup g_\xi) \models_{Q(Q, I, M, S)} \" y \in A \". \]

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References